Training Exam

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

Test Exam, distributed 21 May 2014

Name:

UvA Student ID:

General comments.

1. **The time for this exam is 3 hours (180 minutes).**
2. **There are 72 points in the exam: 36 points are sufficient for passing.**
3. **Please mark the answers to the questions in Exercise I on this sheet by crosses.**
4. **Make sure that you have your name and student ID on each of the sheets you are handing in.**
5. **If you have any questions, please indicate this silently and someone will come to you. Answers to questions that are relevant for anyone will be announced publicly.**
6. **No talking during the exam.**
7. **Cell phones must be switched off and stowed.**

Exercise I	(30 points)	**Exercise IV**	(12 points)
Exercise II	(10 points)	**Exercise V**	(10 points)
Exercise III	(10 points)		
		TOTAL	(72 points)
		GRADE	

Exercise I (30 points).

Find the correct answer (2 points each). Each of the questions has only one correct answer. Please pay attention to negations and whether the question asks for "false" or "true". No points are subtracted for wrong answers.

1. One of the following ordinal equalities is false. Which one?

   ☐ **A** $2 \cdot \omega = 3 \cdot \omega$.

   ☐ **B** $3 + \omega + \omega \cdot \omega = \omega + 3 + \omega \cdot \omega$.

   ☐ **C** $\omega \cdot \omega + 3 = 3 + \omega \cdot \omega$.

   ☐ **D** $12 \cdot (5 + \omega) = 60 \cdot \omega$.

2. Suppose that λ is a limit cardinal. Which of the following statements is provable in ZFC?

   ☐ **A** $\mathrm{cf}(\lambda) = \aleph_0$.

   ☐ **B** The cardinal λ is regular.

   ☐ **C** $\lambda = \aleph_\lambda$.

   ☐ **D** None of the above.

3. The set $\mathbf{V}_{\omega+1}$ was the $\omega + 1$st level of the von Neumann hierarchy, i.e., $\mathbf{V}_{\omega+1} = \wp(\mathbf{V}_\omega)$. Only one of the following axiom (scheme)s of set theory is false in the structure $(\mathbf{V}_\omega, \in)$. Which one?

   ☐ **A** The Axiom of Infinity.

   ☐ **B** The Power Set Axiom.

   ☐ **C** The Union Axiom.

   ☐ **D** The Axiom Scheme of Separation.

4. The Zermelo numbers are defined by the following recursion: $z_0 := \varnothing$ and $z_{n+1} := \{z_n\}$. One of the following statements is false. Which one?

   ☐ **A** $z_{11} \in z_{12}$.

   ☐ **B** $z_{11} \subseteq z_{12}$.

   ☐ **C** $z_{11} \notin z_{15}$.

   ☐ **D** $z_{11} \nsubseteq z_{15}$.

5. Let κ be a cardinal with $\mathrm{cf}(\kappa) = \aleph_0$. Only one of the following statements is provable. Which one?

   ☐ **A** Every injective function from κ to κ has countable range.

   ☐ **B** The set κ is a countable union of sets of cardinality strictly smaller than κ.

   ☐ **C** The cardinal κ is countable.

   ☐ **D** If $\kappa = \aleph_\xi$, then ξ is a countable ordinal.

6. In ZFC, one of the following statements is equivalent to $2^{\aleph_0} = \aleph_1$. Which one?

   ☐ **A** There is a bijection between the power set of $\mathbb{N}$ and the real numbers.
   ☐ **B** Every uncountable set of real numbers contains a set of cardinality $2^{\aleph_0}$.
   ☐ **C** Every set that is equinumerous to the real numbers is uncountable.
   ☐ **D** Every uncountable set has cardinality $2^{\aleph_0}$.

7. Consider the following graph model M:

   One of the following axiom (scheme)s of set theory is false in M. Which one?

   ☐ **A** The Pairing Axiom.
   ☐ **B** The Power Set Axiom.
   ☐ **C** The Union Axiom.
   ☐ **D** The Axiom Scheme of Separation.

8. Let κ, λ, and μ be cardinals. One of the following statements is not generally true. Which one?

   ☐ **A** $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.
   ☐ **B** $\kappa \leq \lambda$ implies $\kappa^\mu \leq \lambda^\mu$.
   ☐ **C** $\kappa \leq \lambda$ implies $\mu^\kappa \leq \mu^\lambda$.
   ☐ **D** $\kappa < \lambda$ implies $\mu^\kappa < \mu^\lambda$.

9. The statement "there are no cardinal numbers between $\aleph_0$ and $\aleph_1$" is...

   ☐ **A** ...provable in ZF,
   ☐ **B** ...provable in ZFC, but not in ZF,
   ☐ **C** ...equivalent to the Continuum Hypothesis in the base theory ZFC.
   ☐ **D** None of the above.

10. Consider the integers $\mathbb{Z}$ with their natural order $<$ and their natural multiplication $\cdot$. One of the following sets is wellordered by $<$. Which one?

   ☐ **A** $\mathbb{Z}\backslash\{0\}$,
   ☐ **B** $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,
   ☐ **C** $\{z \in \mathbb{Z}\,;\, z < 0 \wedge \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,
   ☐ **D** $\{z \in \mathbb{Z}\,;\, \exists x \in \mathbb{Z}(z = x \cdot x)\}$.

11. Let $\aleph(X)$ be the Hartogs aleph for the set X, i.e., the least ordinal α such that $\alpha \not\preceq X$. Only one of the following statements is provable in ZFC. Which one?

☐ **A** $\aleph(\aleph_2) = \aleph_{\aleph_2}$.

☐ **B** $\aleph(\aleph_2) = \aleph_3$.

☐ **C** $\aleph(\aleph_3) \geq 2^{\aleph_2}$.

☐ **D** $\aleph(\aleph_3) = \aleph_2$.

12. Assume AC. One of the following cardinal numbers is *not* regular, which one?

☐ **A** $\aleph_{\aleph_1+4}$.

☐ **B** $\aleph_5$.

☐ **C** $\aleph_{\aleph_5}$.

☐ **D** $\aleph_{\aleph_2+3}$.

13. We often use informal mathematical notation using curly braces to denote sets. However, not every expression corresponds to a set; sometimes, we denote proper classes. Among the following expressions, one corresponds to a proper class. Which one?

☐ **A** $\{x\ ;\ x$ is a nonempty subset of the natural numbers$\}$.

☐ **B** $\{x\ ;\ x$ is a finite set of real numbers$\}$.

☐ **C** $\{x\ ;\ x$ is a one-element set of rational numbers$\}$.

☐ **D** $\{x\ ;\ x$ is a two-element subset of a vector space$\}$.

14. An ordinal γ was called a *gamma number* if for all $\alpha, \beta \in \gamma$, we have that $\alpha + \beta \in \gamma$. Only one of the following ordinals is a gamma number. Which one?

☐ **A** $\omega + \omega$,

☐ **B** $\omega + \omega \cdot \omega$,

☐ **C** $\omega \cdot \omega + \omega$,

☐ **D** $(\omega \cdot \omega) \cdot 2$.

15. An ordinal operation F is a class assigning an ordinal $F(\alpha)$ to each ordinal α. It is called *strictly monotone* if for all $\alpha < \beta$, we have $F(\alpha) < F(\beta)$. It is called *continuous* if for limit ordinals $\lambda > 0$, we have $F(\lambda) = \bigcup\{F(\alpha)\,;\, \alpha < \lambda\}$. An ordinal γ is called a *fixed point of* F if $F(\gamma) = \gamma$. Only one of the following statements is **not** provable for a strictly monotone and continuous ordinal operation. Which one?

☐ **A** There is a fixed point $\gamma \geq \aleph_{\aleph_1}$.

☐ **B** There is a fixed point of cofinality $\aleph_8$.

☐ **C** There are infinitely many fixed points.

☐ **D** There is a regular fixed point.

Exercise II (10 points). Work in Zermelo set theory (i.e., FST plus the Axiom of Infinity; do not use the Axiom Scheme of Replacement!). Prove the following version of the recursion theorem:

> Let X be a nonempty set and let S be the set of finite X-sequences. Let $f : S \to X$ be a function. Then there is a unique function $g : \mathbb{N} \to X$ such that for all $n \in \mathrm{N}$, $g(n) = f(g{\restriction}n)$.

Exercise III (10 points). We call an initial ordinal κ an *aleph fixed point* if $\aleph_\kappa = \kappa$. Show that aleph fixed points exist and that if κ_0 is the smallest one, then $\mathrm{cf}(\kappa_0) = \aleph_0$.

Exercise IV (12 points). Consider the following statement:

> If X and Y are sets and there is a surjection $g : Y \to X$, then there is an injection $f : X \to Y$ such that for all $x \in X$, we have $g(f(x)) = x$. (ESHS)

Prove in ZF^- that (ESHS) is equivalent to the Axiom of Choice.

Exercise V (10 points). Prove the following *fundamental theorem of ordinal division*: Suppose $0 < \alpha < \beta$ are ordinals. Then there are unique $\mu \leq \beta$ and $\varrho < \alpha$ such that $\beta = \alpha \cdot \mu + \varrho$ (ordinal addition and multiplication).