Homework Set #6

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.juli (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.
This homework set is due on **Wednesday 14 May 2014, before the morning *werkcollege*.**

1. As in the lecture, a formula Φ (possibly with parameters) is called an *ordinal operation* if for all ordinals α there is a unique ordinal β such that $\Phi(\alpha, \beta)$. As in the lecture, we write $F_\Phi(\alpha)$ for the unique β such that $\Phi(\alpha, \beta)$.

   We call an ordinal operation *strictly monotone* if for all ordinals α and β, if $\alpha \in \beta$ then $F_\Phi(\alpha) \in F_\Phi(\beta)$. We call it *continuous* if for all limit ordinals λ, we have that
   $$F_\Phi(\lambda) = \bigcup\{F_\Phi(\alpha)\,;\, \alpha \in \lambda\}.$$
   We say that ξ is a *fixed point* of Φ if $F_\Phi(\xi) = \xi$.

   In this entire exercise, the axiomatic setting is ZF^-, i.e., Zermelo set theory plus Replacement.

   (a) Show that every strictly monotone and continuous ordinal operation has arbitrarily large fixed points (i.e., for every α there is a ξ such that $\alpha \in \xi$ and ξ is a fixed point. (**Two useful intermediate steps:** Show that for all α, we have $\alpha \leq F(\alpha)$; and show that for all sets X of ordinals $F(\bigcup X) = \bigcup\{F(\alpha)\,;\, \alpha \in X\}$.)

   (b) We call γ a *principal number of addition* (or *gamma number*) if for all $\alpha, \beta \in \gamma$, we have that $\alpha + \beta \in \gamma$. Show that there is a proper class of gamma numbers.

   (c) We call an initial ordinal κ an *aleph fixed point* if $\aleph_\kappa = \kappa$. We say that an initial ordinal κ *has countable cofinality* if there is a countable $C \subseteq \kappa$ such that $\kappa = \bigcup C$. Show that aleph fixed points exist and that the smallest one has countable cofinality.

2. Work in ZF^-, i.e., Zermelo set theory plus Replacement **without** the Axiom of Choice. Let α be an ordinal. Define the following order $<_{\mathrm{G}}$ on $\alpha \times \alpha$:
   $$\begin{aligned}(\gamma, \delta) <_{\mathrm{G}} (\gamma', \delta') :\Longleftrightarrow\ & \max(\gamma, \delta) < \max(\gamma', \delta') \text{ or}\\ & \max(\gamma, \delta) = \max(\gamma', \delta') \text{ and } \gamma < \gamma' \text{ or}\\ & \max(\gamma, \delta) = \max(\gamma', \delta') \text{ and } \gamma = \gamma' \text{ and } \delta < \delta'.\end{aligned}$$

   (a) Prove that $(\alpha \times \alpha, <_{\mathrm{G}})$ is a wellorder.

   (b) By (a) and the Fundamental Theorem, there is a unique ordinal ξ_α such that $(\xi_\alpha, \in) \simeq (\alpha \times \alpha, <_{\mathrm{G}})$. Prove that if α is an infinite initial ordinal, then $\alpha = \xi_\alpha$. (**Hint.** Use induction on the class of initial ordinals.) Use this to prove *Hessenberg's Theorem*: for any infinite initial ordinal α, we have that $\alpha \sim \alpha \times \alpha$.

3. Work in ZF^-, i.e., Zermelo set theory plus Replacement **without** the Axiom of Choice. Prove that if every set is wellorderable, then the Axiom of Choice holds.