Homework Set #4

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.juli (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.
This homework set is due on **Wednesday 30 April 2014, before the morning *werkcollege*.**

1. Let $\mathbf{X} := (X, <_X)$ and $\mathbf{Y} := (Y, <_Y)$ be strict linear orders. Define the *sum* $\mathbf{X} \oplus \mathbf{Y} := (Z_+, <_+)$ and the *product* $\mathbf{X} \otimes \mathbf{Y} := (Z_\times, <_\times)$ as follows:

$$\begin{aligned} Z_+ &:= X \times \{0\} \cup Y \times \{1\}, \\ Z_\times &:= X \times Y, \\ (z, b) <_+ (z', b') &:\Leftrightarrow (b = 0 \wedge b' = 1) \vee (0 = b = b' \wedge z <_X z') \vee (1 = b = b' \wedge z <_Y z'), \text{ and} \\ (x, y) <_\times (x', y') &:\Leftrightarrow y <_Y y' \vee (y = y' \wedge x <_X x'). \end{aligned}$$

   Prove

   (a) that $\mathbf{X} \oplus \mathbf{Y}$ and $\mathbf{X} \otimes \mathbf{Y}$ are strict linear orders,

   (b) that if $\mathbf{X}$ and $\mathbf{Y}$ are wellfounded, then so are $\mathbf{X} \oplus \mathbf{Y}$ and $\mathbf{X} \otimes \mathbf{Y}$.

2. If $(X, <)$ is a strict linear order, then a function $s : X \to X$ is called a *successor function* if for all $x \in X$, we have $x < s(x)$ and there is no x' such that $x < x' < s(x)$. We call an element $m \in X$ a *smallest element* if for all $x \in X$, $m < x$ or $m = x$. If $(X, <)$ is a strict linear order with smallest element m and successor function s, we say that it satisfies *complete induction* if

$$\forall A \subseteq X \left((m \in A \wedge \forall x (x \in A \to s(x) \in A)) \to A = X \right).$$

   (a) Assume that $\mathbf{X} := (X, <_X)$ and $\mathbf{Y} := (Y, <_Y)$ are strict linear orders with successor functions s_X and s_Y. Prove that both $\mathbf{X} \oplus \mathbf{Y}$ and $\mathbf{X} \otimes \mathbf{Y}$ have a successor function.

   (b) Show that there are strict linear orders with smallest element and successor function that do not satisfy complete induction.

3. Prove (in ZF^-) the most general version of the Recursion Theorem:

   Let φ be a formula in $n+2$ variables. Let $p_0, ..., p_{n-1}$ and X be sets and suppose that (X, R) is a wellfounded relation. If $x \in X$, we denote by $\mathrm{Pr}_R(x) := \{y \in X \,;\, yRx\}$ the set of R-predecessors of x. Suppose that

$$\forall x \exists y \varphi(x, y, p_0, ..., p_{n-1}) \text{ and}$$

$$\forall x \forall y \forall y' \varphi(x, y, p_0, ..., p_{n-1}) \wedge \varphi(x, y', p_0, ..., p_{n-1}) \to y = y'.$$

   Then there is a function f with $\mathrm{dom}(f) = X$ such that for all $x \in X$, we have

$$f(x) = z \iff \varphi(f{\restriction}\mathrm{Pr}_R(x), z, p_0, ..., p_{n-1}).$$