Homework Set #3

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.juli (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.

This homework set is due on **Wednesday 23 April 2014, before the morning *werkcollege*.**

1. We defined the set $\mathbb{N}$ of natural numbers as the minimal inductive set and proved that it satisfies the *Induction Axiom* $(A \subseteq \mathbb{N} \wedge \varnothing \in A \wedge \forall x(x \in A \to x \cup \{x\} \in A) \to A = \mathbb{N})$. Prove the following statements about $\mathbb{N}$:

   (a) We define binary relations R and S on $\mathbb{N}$ by $R(x,y) :\iff x \subseteq y$ and $S(x,y) :\iff x \in y$. Show that $(\mathbb{N}, S)$ is a strict linear order (i.e., irreflexive, transitive and linear) and that $(\mathbb{N}, R)$ is a linear order (i.e., reflexive, transitive, anti-symmetric, and linear) and that

   $$R(x,y) \leftrightarrow S(x,y) \vee x = y.$$

   (b) For any two different natural numbers $n \neq m$, there is no bijection between them.

2. We defined the following formula

   $$\Phi_{\text{plus}}(x,y,z) :\iff \exists a \exists b \exists f \exists g (a \cap b = \varnothing \wedge a \cup b = z \wedge \text{Bij}(f,x,a) \wedge \text{Bij}(g,v,b))$$

   where $\text{Bij}(h,v,w)$ is an abbreviation for the formula expressing "h is a bijection from v to w". Show that this formula defines a binary function on $\mathbb{N}$, i.e., that for every $x, y \in \mathbb{N}$ there is a unique $z \in \mathbb{N}$ such that $\Phi(x,y,z)$ holds.

3. We reconstruct parts of ordinary mathematics in our formal setting. Please work very carefully in the mentioned axiomatic framework and state *explicitly* which axioms you are using in which step. Show that in FST, we can define the notions of *field* and *vector space* (i.e., there is an $\mathcal{L}_\in$-formula describing the statement "V is a K-vector space").

4. Prove that (Ext)+(Pow)+(Sep)+(Repl) proves (Pair).