Homework Set #2

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.julia (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.

This homework set is due on **Wednesday 16 April 2014, before the morning *werkcollege*.**

1. Remember the "singleton axiom" and the "binary union axiom":

$$\forall x \exists s \forall z (z \in s \leftrightarrow z = x) \tag{Sing}$$

$$\forall x \forall y \exists u \forall z (z \in u \leftrightarrow z \in x \vee z \in y) \tag{BinUn}$$

   Consider the following axiom systems:

$$\begin{aligned} &T_0 := \text{(Ext)}+\text{(Sep)}, \\ &T_1 := T_0 + \text{(Un)}+\text{(Pair)}, \\ &T_2 := T_0 + \text{(Sing)}+\text{(BinUn), and} \end{aligned}$$

   (a) Show that T_1 implies the validity of all axioms in T_2.

   (b) Show that T_2 implies (Pair).

2. Consider two disjoint copies of the model H_∞ of the hereditarily finite sets constructed in class and call them $H_{\infty,0} = (V_{\infty,0}, E_{\infty,0})$ and $H_{\infty,1} = (V_{\infty,1}, E_{\infty,1})$. Each of the two models has a unique bottom vertex (the empty set) which we shall call e_0 and e_1, respectively. Let e be a vertex that doesn't occur in either $V_{\infty,0}$ or $V_{\infty,1}$. We construct a new model $M = (V, E)$ as follows:

   - $V := \{e\} \cup V_{\infty,0} \backslash \{e_0\} \cup V_{\infty,1} \backslash \{e_1\}$;
   - if $x \in V_{\infty,i}$, then eEx iff $e_i E_{\infty,i} x$;
   - for any $x, y \in V_{\infty,i}$, we let xEy iff $xE_{\infty,i}y$;
   - if $x \in V_{\infty,0}$ and $y \in V_{\infty,1}$, then neither xEy nor yEx holds.

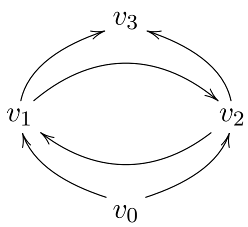
   Show that M satisfies (Un), (Pow), (Sing), and (Sep), but not (Ext), (Pair), and (BinUn).

3. The *Hausdorff definition of the ordered pair* was $(a, b)_{\mathrm{H}} := \{\{a, \varnothing\}, \{b, \{\varnothing\}\}\}$. The *simplified Kuratowski definition of the ordered pair* was $(a, b)_{\mathrm{SK}} := \{a, \{a, b\}\}$. We say that a definition of ordered pairs $(\cdot, \cdot)_\bullet$ is *adequate over a model* $G = (V, E)$ if for all a, a', b, b', we have that

$$(a, b)_\bullet = (a', b')_\bullet \iff a = a' \text{ and } b = b'.$$

   (a) Show that the Hausdorff definition $(\cdot, \cdot)_{\mathrm{H}}$ is adequate over every model of FST.

   (b) Consider the following model $G = (V, E)$



   and find pairwise different vertices a, b, and c such that $(a, b)_{\mathrm{SK}} = (c, b)_{\mathrm{SK}}$ (note that this in particular means that both $(a, b)_{\mathrm{SK}}$ and $(c, b)_{\mathrm{SK}}$ have to be defined). In other words, $(\cdot, \cdot)_{\mathrm{SK}}$ is not adequate over G.

   (c) The model G mentioned in (b) satisfies (Ext), but not (Pair) or (Sep). Expand the model G to a model of (Ext), (Sep) and (Pair), that still contains G as a subgraph and witnesses that $(\cdot, \cdot)_{\mathrm{SK}}$ is not adequate.

   (*Hint.* Follow the idea of the construction of G_∞ from class.)