Handout: Kőnig's Lemma

Axiomatische Verzamelingentheorie
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Universiteit van Amsterdam

http://hugonobrega.github.io/teaching/AxVT/

Theorem (Kőnig's Lemma). For every cardinal κ, we have $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.

Proof. We write $\lambda := \mathrm{cf}(\kappa) \leq \kappa$. Fix some cofinal function $g : \lambda \to \kappa$. If $\lambda = \kappa$, the claim follows from Cantor's theorem. Therefore, we can assume that $\lambda < \kappa$. We observe that by definition, $\kappa^\lambda \sim \{f\,;\, f : \lambda \to \kappa\}$, so it is sufficient to show that there cannot be any surjection from κ onto the set $F := \{f\,;\, f : \lambda \to \kappa\}$.

Suppose that $\pi : \kappa \to F$ is an arbitrary function. For $\alpha \in \kappa$, we consider $\pi(\alpha) : \lambda \to \kappa$. Clearly, for every $\alpha < \kappa$, we have that $\mathrm{ran}(\pi(\alpha)) \preceq \lambda$. Therefore, for every $\alpha < \kappa$, we have that

$$R_\alpha := \bigcup\{\mathrm{ran}(\pi(\xi))\,;\, \xi < \alpha\} \preceq \lambda \times \alpha \prec \kappa.$$

So, $\kappa \backslash R_\alpha \neq \varnothing$. We define a *diagonal function*

$$d : \lambda \to \kappa : \beta \mapsto \min(\kappa \backslash R_{g(\beta)}).$$

We claim that $d \notin \mathrm{ran}(\pi)$. Fix an arbitrary $\alpha < \kappa$, and show that $d \neq \pi(\alpha)$. Since g was cofinal, we find $\beta \in \lambda$ such that $g(\beta) > \alpha$. Then, by definition,

$$d(\beta) \notin R_{g(\beta)} = \bigcup\{\mathrm{ran}(\pi(\xi))\,;\, \xi < g(\beta)\} \supseteq \mathrm{ran}(\pi(\alpha)).$$

So, in particular, $d(\beta) \neq \pi(\alpha)(\beta)$, and so $d \neq \pi(\alpha)$. q.e.d.

Comments.

1. Normally, this is proved in a more general form. If $\sum\{X_\alpha\,;\, \alpha < \gamma\}$ is the disjoint union of the sets X_α and $\prod\{X_\alpha\,;\, \alpha < \gamma\} := \{f\,;\, \mathrm{dom}(f) = \gamma \text{ and for all } \alpha \in \gamma\text{, we have that } f(\alpha) \in X_\alpha\}$, then if for all $\beta < \lambda$, we have $X_\beta \prec Y_\beta$, then

$$\sum\{X_\beta\,;\, \beta < \lambda\} \prec \prod\{Y_\beta\,;\, \beta < \lambda\}.$$

   [How can one see that the above theorem is a special case of this statement? Consider $X_\beta := g(\beta)\backslash \bigcup\{g(\eta)\,;\, \eta < \beta\}$ and $Y_\beta := \kappa$.]

2. Not to be confused with another "Kőnig's Lemma" which is a statement about existence of infinite branches in finitely branching trees. Different theorem, different —but related— Kőnig: our Kőnig's Lemma was proved by Gyula Kőnig (1849–1913) who also published under the name "Julius König"; the statement about trees was proved by his son Dénes Kőnig (1884–1944).