Homework Set #5

Axiomatische Verzamelingentheorie
2013/14: 2nd Semester; block b
Universiteit van Amsterdam

`http://hugonobrega.github.io/teaching/AxVT/`

The *preferred method* for submitting homework solutions is by handing them in before the start of the *werkcollege* on Wednesday morning. Electronic submissions are also possible, by email to `ilin.juli (at) gmail.com` or `hugonobrega (at) gmail.com` before the deadline. These must be in a single, legible PDF file. It is also possible to hand in your homework by putting it into Julia's or Hugo's mailboxes at the ILLC at Science Park 107, but in this case the deadline for submission is at 10:45am on the day of the submission deadline (before Julia and Hugo leave for the *werkcollege*). It is important to respect the strict deadlines stated above; late homework will not be accepted.
This homework set is due on **Wednesday 7 May 2014, before the morning *werkcollege*.**

1. If $(X, <)$ is a strict linear order, we define the *order topology* as follows: for $x \leq y$, we let $\text{Interval}(x, y) := \{z \in X\,;\, x < z < y\}$ (called the *open interval between x and y*) and we call a set $Z \subseteq X$ *open* if it is a union of open intervals (not necessarily a finite union). Then the *order topology* τ is the set of open sets in X. As usual in topology, a point $x \in X$ is called a *τ-limit point* if for every $U \in \tau$ with $x \in U$ there is some $y \neq x$ such that $y \in U$.

   Remember from homework set #4 that we called a function $s : X \to X$ is a *successor function* if for all $x \in X$, we have $x < s(x)$ and there is no x' such that $x < x' < s(x)$. The elements of $\text{ran}(s)$ are called *s-successors*, the elements of $X \backslash \text{ran}(s)$ are called *s-limits*.

   (a) Prove that if $(X, <)$ is a strict wellorder with order topology τ and a successor function s, then a point x is a τ-limit point if and only if it is an s-limit.

   (b) Prove that if $(X, <)$ is a wellorder, then it has a unique successor function.

2. An ordinal α is called an *initial ordinal* if for all $\beta \in \alpha$, we have $\beta \not\sim \alpha$. Show that for every initial ordinal α at least one of the following three cases holds:

   Case 1. $\alpha = \varnothing$.

   Case 2. There is an ordinal $\beta \in \alpha$ such that $\alpha = \aleph(\beta)$.

   Case 3. There is a set X of ordinals such that $\alpha = \bigcup\{\aleph(\beta)\,;\,\beta \in X\}$.

3. Let X be an arbitrary set. We consider functions $H : \wp(X) \to \wp(X)$. Such a function is called *monotone* if for all $A \subseteq B$ implies $H(A) \subseteq H(B)$. We say that $T \subseteq X$ is a *fixed point of H* if $H(T) = T$. As usual, if f is a function and $A \subseteq \text{dom}(f)$, we write $f[A] := \text{ran}(f{\restriction}A)$.

   (a) Show that every monotone H has a fixed point. [**Hint.** Consider the set $\bigcup\{A \subseteq X\,;\, A \subseteq H(A)\}$.]

   (b) Consider two sets X and Y such that there are injections $f : X \to Y$ and $g : Y \to X$. Define the function $H : \wp(X) \to \wp(X)$ by
   $$H(A) := X \backslash g[Y \backslash f[A]].$$
   Show that H is monotone (and thus has a fixed point).

   (c) Prove the *Cantor-Schröder-Bernstein Theorem*: if $X \preceq Y$ and $Y \preceq X$, then $X \sim Y$. [**Hint.** Let $f : X \to Y$ and $g : Y \to X$ be the injections. Use a fixed point T of the function H given in (b) and define the bijection on T by f and on $X \backslash T$ by g^{-1}.]